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The Landauer model for electrical resistance: its extension from quantum to classical transport

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Abstract. Problems in the Landauer model for electrical resistivity are conventionally approached using transmission (T -) matrix methods. Using a group theoretic approach it is shown that a transformation from the usual spinorial ($SU(1, 1) \cup Sp(2, R)$ or $SL(2, R)$) to a vectorial ($SO(2, 1)$) representation allows successive regions to be considered as either successive non-Euclidean geometric operations or boosts. By borrowing a T -matrix method from the model's optical analogue, that of propagation in plane-parallel (Fabry–Perot) media, an exact treatment of a regular (Kronig–Penney) lattice is developed, showing that transport is via delocalised states with vanishing resistance as expected. Using the analogue to elucidate the coherent multiple scattering nature of the Landauer model, it is seen that the model's validity is restricted to mesoscopic samples, i.e. those of length $L < l_\phi$ where l_ϕ is the coherence length. Thus previous problems in calculating average resistances using Landauer models are associated with 'universal fluctuations' and quantum interference effects, with coherent propagation of the wavefunction as their origin. A proper treatment of incoherent scattering, valid for samples considered to be composed of units of length $L > l_\phi$, gives classical (Ohmic), additive scaling of resistance, contrary to an oft-quoted scaling theory of resistance and localisation.

1. Introduction

In 1957 Landauer proposed a one-dimensional model for electrical resistance in random alloys, whereby resistance is due to scattering from the static disorder [1]. The model's simplicity and the curious nature of the results, namely an exponential length dependence of the resistance, have led to a considerable volume of studies based on this formalism. Attempts have been made to generalise the theory to higher-dimensional samples [2–5], to link this non-Ohmic nature with (Anderson) localisation [6, 7], and numerical studies have been used to confirm the length dependence [8]. Being in real space, the model is more easily visualised than the traditional reciprocal-space models and is unique in regarding the current as the driving force, which in turn leads to the potential difference required to maintain this current [9].

The Landauer model is based on solutions to the one-dimensional, time-independent Schrödinger equation in a single-electron approximation, so that a pseudo-potential representing both interactions with the lattice and the remaining electrons is used. Transmission (T -) matrix methods are a powerful tool which enable the construction of solutions to the Schrödinger equation in a composite region from solutions for the component regions. By considering the group theoretical properties of T -matrices

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it is shown that one can move from the usual 2×2 spinorial to a 4×4 vectorial representation allowing new insight into the composition of products of T -matrices.

Another area in which T -matrices have been used is in the design of thin-film optical devices. Such Fabry–Perot problems are an optical analogue and are seen to provide two valuable insights into the limitations of the Landauer model. Firstly the Landauer model is seen to be limited to multiple coherent scattering, whereas it is well known that incoherent scattering is intrinsic to conventional electron transport models. Secondly, in the same way that Fabry–Perot devices are a limited set of possible optical structures, it is seen that the Landauer model is limited, for realisable materials, to composites of a plane parallel nature.

Using an old method for solutions to optical propagation in a periodic ab -type multilayer, it is shown that the archetypal problem of an electron in a periodic potential can be solved exactly. Furthermore, this solution shows the expected behaviour that transport is via delocalised states with a vanishing resistance.

By relating the T -matrix algebra to a multiple scattering formalism, it is shown how incoherent scattering may be included, though this is at the expense of the T -matrix algebra. A previous result for incoherent scatterers, central to scaling theories of resistance and localisation [10], is shown to be in error and pure incoherent scattering leads to Ohmic, linear scaling of resistance with length.

The coherence length l_ϕ at which the wavefunction's phase becomes random is seen to be of central importance. Constructing a conductor from units of length $L > l_\phi$ allows one to use Ohm's law to calculate the overall resistance from the individual resistances. For conductors of length $L < l_\phi$ coherence is important [11–13] and previous Landauer-type results are viable. Thus, two curious results of the Landauer model are explained. The first, the difficulty in defining an average resistance for a representative sample [14], is associated with the non-commutative nature of the transmission matrices, and has led to many novel approaches, such as defining 'typical' rather than average resistances [15] and calculating the higher moments of the resistance [16, 17]. The second, is the ability to construct a sandwich sample of the form ABA and, through the proper choice of B, ensure that the overall resistance of the structure vanishes, regardless of how great the resistance of sample A [18]. Both of these effects are unphysical for macroscopic samples, but do occur on sufficiently small length scales where the coherent Landauer model applies, the former being associated with the 'universal fluctuations' [19, 20] in the resistance of small-scale samples, and the latter being one of the phenomena associated with multiple quantum well semiconductor structures [21].

Thus it is seen that the Landauer model is adequate for small samples exhibiting ballistic transport, but that at large scales, the dominant, incoherent scattering invalidates the model.

2. The Landauer model

The starting point for the Landauer model is the time-independent Schrödinger equation

$$(\hbar^2/2m)(d^2\psi/dx^2) = (V(x) - E) \psi \quad (1)$$

where $V(x)$ represents the pseudo-potential due to the lattice and the remaining electrons. Whilst the standard semi-classical methods start with a periodic $V(x)$,

Landauer's original paper used this equation as the starting point for an analysis of a spatially disordered conductor whence $V(x)$ is assumed to be aperiodic.

In this section the tools which have been used to investigate equation (1) will be considered, since this gives considerable insight into the physics of the Landauer model.

2.1. Transmission matrix methods

The spatial evolution of the solution to equation (1) in regions where the potential $V(x)$ is discontinuous is followed by ensuring the continuity of the wavefunction and its first derivative. This is conveniently and concisely handled through the introduction of transmission (T -) matrices (solutions in the WKB approximation can similarly be traced through T -matrices). There are a host of equivalent representations for the T -matrices, the two most natural being that (type-1) [22, 23] relating the vectors

$$\psi(x) = \begin{pmatrix} \psi \\ \psi' \end{pmatrix}$$

on either side of a discontinuity in the potential, where ψ' denotes the spatial derivative, whilst the second, spinor representation (type-2) [24] is related to the first by $\phi = \mathbf{U}\psi$ and connects vectors

$$\phi(x) = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} = \begin{pmatrix} 1/2 & -i/2k \\ i/2k & 1/2 \end{pmatrix} \begin{pmatrix} \psi \\ \psi' \end{pmatrix}$$

where ψ^\pm , up to a phase, denotes the amplitude of the forward/backward travelling component in a region with wavevector k .

In both cases the determinant of the transmission matrix is one, a requirement resulting from the restriction to a real potential function, $V(x)$, ensuring that the probability current density, Wronksian, and hence electrical current in the Landauer model, is conserved [15]. Further restrictions on the relationship between the components of a type-2 T -matrix may be obtained through symmetry considerations as described by Merzbacher [25].

For an arbitrary real, symmetric barrier centred at the origin, the T -matrix has the form

$$\phi(x) = \begin{pmatrix} 1/t^* & r/t \\ r^*/t^* & 1/t \end{pmatrix} \phi(-x) \tag{2}$$

where t and r are the transmission and reflection amplitudes, perhaps more commonly seen in the S -matrix [25, 26]:

$$\eta_{\text{out}} = \begin{pmatrix} t^* & r^* \\ r & t \end{pmatrix} \eta_{\text{in}}. \tag{3}$$

Within the Landauer model, the relation between the resistance and microscopic parameters is [1, 27]

$$R = (2\pi\hbar/e^2)\rho$$

where ρ is a dimensionless resistance, given by the ratio of the reflection (albedo) and transmission probabilities, $|r|^2/|t|^2$. This basic equation has been extended and verified

by numerous authors [28, 29], a particularly clear discussion of its relationship with a classical picture being in the review by Erdős and Herndon [14].

In principle then, with knowledge of the potential as a function of x , the individual T -matrices and hence the overall T -matrix for the region to be studied can be calculated; the resistance is then obtained directly from the off-diagonal elements. Of course for a realistic material, represented by a series of discontinuities in the potential, the overall T -matrix is a product of a very large number of matrices, so that numerical approximations are commonly made.

2.2. Group theoretical considerations

A recent embellishment to the discussion of T -matrix methods has been their classification in group theoretical terms [18], the restriction to a particular group being an embodiment of the symmetries inherent in the solutions to equation (1). Thus it can be shown that, for *any* region: its type-2 matrix belongs to either $SU(1, 1)$ or $Sp(2, R)$, whilst its type-1 matrix belongs to $SL(2, R)$. This information is sufficient as a basis of a random matrix approach [30], though some assumption has to be made as to the detailed distribution of the matrices. Perhaps more important is that the 2×2 matrices form the basis of a projective representation of the special orthogonal group in $(2+1)$ dimensions, $SO(2, 1)$ (a particularly transparent discussion of the relationship between $SO(2, 1)$ and its associated 2×2 matrix representations is included in the review by Balazs and Voros [31]). This allows a move from a spinorial to a vectorial representation, in an exactly analogous fashion to the correspondence between the Jones ($SU(2)$) and Mueller ($SO(3)$) matrix representations of geometrical optics [32], vectors and operations in the orthogonal representation being more readily visualised. A simple discussion of the relationship between spinorial and vectorial representations can be found in [33].

The path to the 4×4 representation can be seen by decomposing the density matrix (corresponding to the coherency matrix of Jones calculus) using the Pauli spin matrices [34]

$$\psi\psi^\dagger = \begin{pmatrix} |\psi|^2 & \psi\psi'^* \\ \psi'^*\psi & |\psi'|^2 \end{pmatrix} = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$$

to form a null 4-vector, $x^\mu = (t, x, y, z)$, where the appropriate components are

$$\begin{aligned} t &\mapsto \frac{1}{2}(|\psi|^2 + |\psi'|^2) \\ x &\mapsto \frac{1}{2}(\psi\psi'^* + \psi'^*\psi) \\ y &\mapsto \frac{1}{2}i(\psi\psi'^* - \psi'^*\psi) \\ z &\mapsto \frac{1}{2}(|\psi|^2 - |\psi'|^2). \end{aligned}$$

For the corresponding decomposition in the case of the coherency matrix, these four real quantities are the only directly measurable quantities, the Stokes parameters, whilst the corresponding 4-vector is the Stokes vector.

The equality, $t^2 = x^2 + y^2 + z^2$ ensures the null quality of the vector, whilst the invariance of the Wronskian of equation (1), corresponding to the y -component or more physically the electrical current in the Landauer model, means that this equality can be written as $t^2 - x^2 - z^2 = \text{const} = I^2$. It can now be seen that all solutions lie on the surface of an hyperboloid, so that any T -matrix connects two points on this

surface and hence can be considered to be a Minkowskian (Lobachevskian–Bolyai) geometrical operation [35, 36]. (The geometrical nature of $SU(1, 1)$ mappings has recently been discussed [37].) Thus, the overall T -matrix (4×4) for a series of regions can be described as the geometrical operation which is the resultant of the successive application of the geometrical operations corresponding to the T -matrices (4×4) for the component regions.

Whilst general aspects of the orthogonal group mapping have been noted elsewhere, the decomposition of regions of constant potential has yet to be considered. As an example the type-2 T -matrix for a region of length b and wavevector K , equation (6), can be decomposed;

$$\mathbf{B} = \begin{pmatrix} K^{-1/2} & 0 \\ 0 & K^{1/2} \end{pmatrix} \begin{pmatrix} \cos Kb & \sin Kb \\ -\sin Kb & \cos Kb \end{pmatrix} \begin{pmatrix} K^{1/2} & 0 \\ 0 & K^{-1/2} \end{pmatrix}$$

and corresponds to a boost of rapidity $\ln K$ in the z direction, followed by a rotation of angle $-Kb/2$ about the y axis, and finally a boost of rapidity $\ln K$ in the $-z$ direction: for imaginary K , the rotation above is replaced by a boost of rapidity $Kb/2$ in the x direction. Thus, it is possible to analyse a discontinuous series of constant potentials in terms of products of boosts along the x and z directions and rotations about the y axis (the hyperboloid surface lies in the x, z, t space). The algebra of boosts and rotations is well documented [38], so this may prove to be a useful representation for Landauer-type problems.

2.3. The optical analogue

The Schrödinger equation in one dimension has two analogues [39]: the electrical transmission line, and optical Fabry–Perot (FP) problems [40]. The FP analogue of a lattice of the Kronig–Penney type is a series of parallel optical plates, where the electron wavevector is replaced by the s -component of a normally incident, coherent, light beam [41]. Many results can be carried over directly between the different systems (transmission line analogues are frequently used in discussing FP systems and vice versa [42, 43]), the reflection and transmission amplitudes being replaced by the complex Fresnel coefficients. Matrix methods for FP problems have a long history [44], and have been extensively used to predict properties for multilayer film devices: for instance, the type-1 matrices are entirely analogous to the Herpin matrix representation [45], whilst a type-2 representation may be traced to Abelès' work [46].

Since in the FP problems the optical wave is treated as being coherently propagated, the analogy then implies that the electron wavefunction is also being coherently propagated. The correct treatment of incoherent scattering and the multiple scattering nature of the Landauer model will be treated in a later section.

It is also obvious from the analogue that realistic (three-dimensional) materials which exhibit the Landauer behaviour correspond to plane parallel devices. In the same way that the optics of planar stratified materials is removed from that of particulate media, one may expect that the Landauer model is restricted to an unrepresentative class of materials.

2.4. The Kronig–Penney Lattice

The simplest potential profile to consider is the Kronig–Penney lattice [47] (figure 1), the periodicity of the lattice allowing the overall T -matrix for an n -period sample to

be written [25]

$$\psi(l) = (\mathbf{AB})^n \mathbf{A} \psi(r)$$

or equivalently

$$\phi(l) = (\mathbf{A}'\mathbf{B}')^n \mathbf{A}' \phi(r)$$

where $\psi = \mathbf{U}\phi$ and $\mathbf{A}' = \mathbf{U}\mathbf{A}\mathbf{U}^{-1}$, $\mathbf{B}' = \mathbf{U}\mathbf{B}\mathbf{U}^{-1}$. Many textbooks resort to a qualitative analysis [24, 25], whilst recently a numerical method has been used [48], though the exact result can be obtained either using Chebyshev polynomials [49] or using Sylvester's theorem [50]. Interestingly, the equivalent optical problem was solved some time ago by Abelès [51], where the regions I, II correspond to non-absorbing films of different refractive index.

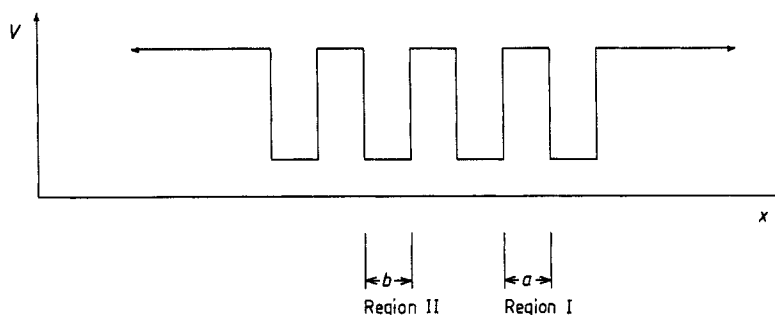


Figure 1. $V(x)$ for a Kronig-Penney lattice.

The appropriate T -matrices are

$$\mathbf{A} = \begin{pmatrix} \cos ka & (\sin ka)/k \\ -k \sin ka & \cos ka \end{pmatrix} \quad (4)$$

$$\mathbf{A}' = \begin{pmatrix} e^{ika} & 0 \\ 0 & e^{-ika} \end{pmatrix} \quad (5)$$

$$\mathbf{B} = \begin{pmatrix} \cos Kb & (\sin Kb)/K \\ -K \sin Kb & \cos Kb \end{pmatrix} \quad (6)$$

$$\mathbf{B}' = \begin{pmatrix} \cos Kb + \frac{1}{2}i(k/K + K/k) \sin Kb & \frac{1}{2}i(k/K - K/k) \sin Kb \\ \frac{1}{2}i(k/K - K/k) \sin Kb & \cos Kb - \frac{1}{2}i(k/K + K/k) \sin Kb \end{pmatrix} \quad (7)$$

where $k = (V(\text{I})/2m)^{1/2}/\hbar$ and $K = (V(\text{II})/2m)^{1/2}/\hbar$. The product, \mathbf{AB} , can be written

$$\begin{pmatrix} \cos \eta + i\beta & \delta - i\gamma \\ \delta + i\gamma & \cos \eta - i\beta \end{pmatrix}$$

where as at all stages in this calculation the trigonometric functions can be replaced by their hyperbolic counterparts as k , K and η become imaginary. With elementary matrix algebra, this matrix can be expressed in terms of its eigenvalues and eigenvectors as

$$\begin{pmatrix} -i \sin \eta - i\beta & i \sin \eta - i\beta \\ -\delta - i\gamma & -\delta - i\gamma \end{pmatrix} \begin{pmatrix} e^{i\eta} & 0 \\ 0 & e^{-i\eta} \end{pmatrix} \left[\begin{pmatrix} -i \sin \eta - i\beta & i \sin \eta - i\beta \\ -\delta - i\gamma & -\delta - i\gamma \end{pmatrix} \right]^{-1}$$

and hence the n th power of \mathbf{AB} is

$$\begin{pmatrix} \cos n\eta + i\beta(\sin n\eta)/(\sin \eta) & (\delta - i\gamma)(\sin n\eta)/(\sin \eta) \\ (\delta + i\gamma)(\sin n\eta)/(\sin \eta) & \cos n\eta + i\beta(\sin n\eta)/(\sin \eta) \end{pmatrix}.$$

Since the remaining T -matrix, \mathbf{A} , merely gives an insignificant phase change, the Landauer resistance can be written as

$$\rho_{\text{tot}} = \rho_1(\sin^2 n\eta)/(\sin^2 \eta)$$

where ρ_1 is the single barrier resistance,

$$\rho_1 = \frac{1}{4} (k/K - K/k) \sin^2 Kb.$$

The behaviour of ρ_{tot} can now be separated into two distinct regions according as either the trigonometric functions, or the hyperbolic functions hold. In the hyperbolic case, the now typical Landauer phenomenon is found: the resistance diverges exponentially with n , *but this is to be expected* since this is in a forbidden region. In the allowed (trigonometric) region the resistance varies periodically with n , and deserves closer consideration.

For a particular value of n , and sweeping across a region of energies such that η varies continuously through the allowed region, the resistance *vanishes* $n - 1$ times (The numerical analysis [48] missed the crucial vanishing, and leaves one wondering about the sensitivity of such studies.) This is entirely analogous to the vanishing of the reflection coefficient for particle waves incident on a square well at energies corresponding to the quasi-bound eigenstates [52]. The vanishing points for the periodic lattice then correspond to the quasi-free electron modes: the Bloch waves [53].

In summary it is seen that for an ordered lattice the results must be treated with some care, as the important regions can obviously be overlooked. An exponential scaling of resistance corresponds to evanescent behaviour in the disallowed electron energy bands, whilst in the allowed region delocalised states offer no resistance to electron transport. Thus the Landauer and standard models are entirely in agreement for the regular lattice.

3. Scattering in the Landauer model

In this section, first the relative importance of incoherent and coherent scattering and its length dependence will be elucidated and then, by reference to the optical analogue and its interpretation, the method by which incoherent propagation can be included in a Landauer-type model will be shown.

3.1. The relationship with 'traditional' models

Since the model uses only time-independent methods, it can only be expected to predict results for experiments in which the resistance is due solely to time-independent factors, i.e. where the timescales for the experiment are much greater than the timescales for the resistance giving mechanisms. The residual resistance for disordered alloys (the asymptotic resistance at $T = 0$ K), is due to static lattice disorder in the Hume-Rothery model [54], and hence would seem to fulfill these requirements. The Hume-Rothery results are good for simple, truly random alloys, and give a particularly

simple compositional law [55]. The Landauer model however, has always produced an exponential length dependence, and no compositional law. In both cases scattering is the only source of resistance, but crucially for the Hume-Rothery model scattering is treated as being incoherent (hence Fermi Rule-type summations [56]), whilst in the Landauer model the wavefunction evolves through multiple, coherent scattering. The importance of incoherent scattering is well known [57], its role in Joule heating being self-explanatory.

In the familiar reciprocal space treatments of resistance the starting point is a regular lattice, from which the allowed modes are calculated. These are most simply seen in the Sommerfeld model [58], where they are merely the eigenmodes for the 'particle in a box'. Then residual resistance is due to deviations from regularity as in the Hume-Rothery model. It has already been shown that the Landauer resistance for a regular lattice vanishes periodically in the allowed region, so at this stage the two approaches are consistent. The difference in results for disordered lattices is due to the coherent treatment of scattering in the Landauer model as will be shown later.

3.2. The multiple scattering nature of the Landauer model

It has been seen in §2.1 that for a single barrier the S -matrix representation, equation (3), is equivalent to the T -matrix representation, equation (2), allowing the interpretation of the T -matrix for a sample in terms of its scattering amplitudes. It is this that allows an interpretation of the physical aspects of the T -matrix so here the composition of the overall T -matrix for two regions, in terms of their scattering amplitudes, will be considered.

A sample of a conductor formed from two regions, each described through T -matrices (type-2), gives

$$\begin{pmatrix} 1/t_{\text{tot}}^* & r_{\text{tot}}/t_{\text{tot}} \\ r_{\text{tot}}^*/t_{\text{tot}}^* & 1/t_{\text{tot}} \end{pmatrix} = \begin{pmatrix} 1/t_2^* & r_2/t_2 \\ r_2^*/t_2^* & 1/t_2 \end{pmatrix} \begin{pmatrix} 1/t_1^* & r_1/t_1 \\ r_1^*/t_1^* & 1/t_1 \end{pmatrix} \quad (8)$$

an equation having no direct representation in terms of the individual S -matrices. This is because the equivalent S -matrix form would require all orders of scattering to be included: i.e. the T -matrix result is inherently of a coherent, multiple-scattering nature. Specifically, the total transmission amplitude (8) can be expanded to reveal this multiple scattering nature [59]:

$$t_{\text{tot}}^* = t_1^* t_2^* [1/(1 - r_1^* r_2^* e^{i\phi})] = t_1^* t_2^* (1 + r_1^* r_2^* e^{i\phi} + (r_1^* r_2^* e^{i\phi})^2 + \dots). \quad (9)$$

Thus the overall transmission amplitude is seen to arise from the summation over all possible scattering paths, and hence over all orders of scattering [60]. Interestingly, in optics the multiple scattering result is commonly used to establish the T -matrix representation [44].

Here then, the combination of two scatterers has been expanded in terms of individual scattering events which, certainly from a particle picture, are the basis for a discussion of the overall physics.

3.3. Types of scattering and their length dependence

The possible types of scattering can be placed into three categories: coherent, incoherent and inelastic scattering. Inelastic scattering is necessarily not coherent, but is

meaningless in this model, as best seen by considerations of energy conservation, and will henceforth not be considered.

If equation (1) is rigorously followed incoherent scattering is similarly a meaningless concept, since all obstacles induce the same phase change irrespective of the previous 'history' of the wavefunction. Equation (1) is however an approximation, since a realistic pseudo-potential $V(x)$ will be expected to vary with time. Intuitively the time dependence of the position of the 'lattice sites' will be more pronounced than that of the detailed potential: i.e. lattice vibrations rather than electron-electron interactions will be more important in determining the time dependence of the pseudo-potential. This variation of the inter-site distance will lead to a difference in phase between the different orders of scattering in the expanded form of equation (9), and hence a breakdown of the coherent model and the T -matrix expansion (It has been noted that even the zero-point motion of the lattice is sufficient to dephase the wavefunction after a number of scattering events [61]). For small reflection amplitudes, the summation inherent in the left-hand side of equation (9) will be a reasonable approximation and hence the T -matrix representation will be adequate. At sufficiently short length scales an electron can be considered to have traversed the sample with little or no scattering: transport in this regime is dubbed 'ballistic' [62, 63] and coherent propagation dominates [64].

Interestingly, exactly similar arguments on the phase coherence can be used in discussing Anderson localisation [61, 65]. Since the localisation phenomena is purely coherent [66, 67], arising from the time reversal symmetry of the wavefunction [68], the length over which such localised states can be considered to exist is similarly limited to small scales [69, 70] $L < l_\phi$.

One can then define a crucial length scale (time scale), the incoherence length l_ϕ , after having traversed which the wavefunction loses its phase coherence (this can alternatively be interpreted in terms of a loss of time-reversal symmetry of the wavefunction [61]). Then, the scattering amplitudes for regions of length $L \gg l_\phi$ will have random phase, whilst those representing regions of length $L \ll l_\phi$ will have a fixed phase. Of course both pure incoherent and pure coherent scattering are approximations, but in the limit of large and small scales, respectively they are exact.

3.4. Coherent scattering in a disordered sample

Landauer's original paper considered a lattice where the position of the scatterers is randomly distributed, the analysis of which will be considered here.

Taking the overall resistance directly from equation (8), gives

$$\rho_{\text{tot}} = \rho_1 + \rho_2 + 2\rho_1\rho_2 - (2/|t_1||t_2|^2)\text{Re}(r_1r_2)$$

where $\text{Re}(\)$ denotes the real part. The argument which leads to exponential scaling of the resistance for a disordered sample is that one must average the combined resistance over a range of inter-obstacle distances. Since shifting either the scatterer, or the coordinate origin, causes a linear change in the phase of the reflection coefficient [8, 14], this amounts to an average over the relative phases of r_1 and r_2 , so that the resistance of the sample is

$$\langle \rho_{\text{tot}} \rangle = \langle \rho_1 \rangle + \langle \rho_2 \rangle + 2\langle \rho_1 \rangle \langle \rho_2 \rangle \quad (10)$$

leading to exponential scaling with length [10].

The scaling theory, based on a disordered, coherent scattering sample, leads to a standard deviation of the resistance which increases with length faster than the average resistance [18, 48], a result which has been linked with localisation [10]. This result has led to a number of attempts to define a more representative resistance [14–17] but, as has recently been shown [20], is the explanation of the large fluctuations in the resistance found amongst mesoscopic samples of normal metals [71], where the variance of the conductance is found to be independent of the bulk properties of the sample (hence the terminology ‘universal fluctuations’). This non-classical behaviour is then associated with coherent scattering and hence samples of dimensions $L < l_\phi$ [11–13].

3.5. Incoherent scattering

By definition for incoherent scatterers, *each* order of scattering has a different phase, so the expanded, multiple scattering form of equation (9) is then

$$t_1^* t_2^* [1 + r_1^* r_2^* \exp(\tau_1) + (r_1^* r_2^*)^2 \exp(\tau_2) + \dots + (r_1^* r_2^*)^n \exp(\tau_n)]$$

where the τ_i are related to the inter-obstacle separation [60]. Now, the summation over the various orders of scattering, implicit in the left-hand side of equation (9), and which forms the basis of the T -matrix representation, *cannot* be made. The overall transmission probability for incoherent scattering may however be calculated:

$$|t_{\text{tot}}|^2 = |t_1|^2 |t_2|^2 [1 / (1 - |r_1|^2 |r_2|^2)]$$

leading to the Ohmic, linear scaling relationship

$$\langle \rho_{\text{tot}} \rangle = \langle \rho_1 \rangle + \langle \rho_2 \rangle.$$

In this case the average resistance is well defined [48], and normal macroscopic behaviour is recovered. This Ohmic behaviour is then associated with incoherent scattering, that is with length scales $L \gg l_\phi$.

The classic derivation of a scaling theory based on the Landauer model is by Anderson *et al* [10], and is frequently cited e.g. [2–4, 6–9, 14–18, 28–30, 65]. Their approach is to first calculate the net resistance of a pair of scatterers in series, then calculate the net resistance of a pair of such pairs etc., so that a macroscopic ‘wire’ is built. Correctly, they note that the coherence length is a scale at which; ‘... we might hope to begin to have a universality in the scaling process ...’, and that the phase of the reflection/transmission coefficients are completely uncorrelated at this scale. However, rather than treating this scattering as incoherent, an average is taken over the relative phases as was done in the analysis leading to equation (10). Hence, rather than calculating results for scatterers with random single scattering phases, they have obtained results for scatterers with random ‘ T -matrix’ phases. Thus, their results are applicable to disordered, coherent scatterers, as discussed with regard to equation (10).

It is now obvious that the universal scaling behaviour which begins at the coherence length is the familiar, additive, Ohmic result. Exponential scaling of conductance is then associated with mesoscopic samples ($L < l_\phi$), and linear scaling with macroscopic samples ($L > l_\phi$).

4. Conclusions

Central to the discussion of the Landauer model is the use of transmission (T -) matrices, which allow a compact description of the problem. The Landauer model was conceived as a method of describing electronic properties of disordered lattices, so in § 2.2 group theoretical considerations are used to examine the general nature of solutions to the one-dimensional wave equation. It is pointed out that the symmetry of the wave equation leads to a representation of the problem in terms of the orthogonal group in $(2+1)$ dimensions [18], equivalent to a vectorial representation of the spinorial problem. This leads to a more readily visualised representation of the Landauer model, complete with two new interpretations of the T -matrices. The first is that the action of any region of potential can be considered as a non-Euclidean geometrical operation, so that successive T -matrices correspond to successive geometrical operations. The second, an algebraic representation in terms of boosts and rotations, proves to lead to a particularly simple decomposition of regions of constant potential.

The analogy between the wave equation in one dimension and the propagation of electromagnetic radiation in plane parallel, Fabry–Perot (FP) media [39, 41], described in § 2.3, holds quite generally for the Landauer model, making more transparent the physical nature of the model. Importantly it is obvious that, in the model, propagation of the wavefunction is a multiple scattering, coherent process. The analogue also implies that the Landauer model can be applied to electron propagation in plane parallel media where coherent scattering dominates. Sandwich materials, such as multilayer heterostructure semiconductors [72, 73] and quasi-crystals [74, 75] may then provide a close connection with previous Landauer-type calculations, though the relative importance of coherent and incoherent scattering is currently under debate for these materials [76, 77]. The interest in these devices is due to the novel transmission properties to be expected [76, 78], analogous to those of optical films [79].

By using a T -matrix description of a periodic potential, it has been shown that the overall T -matrix and hence the Landauer resistance may be calculated exactly. Such techniques are known in optics but have not been used for the case of the Kronig–Penney lattice. The result, that the Landauer resistance for such a model vanishes at the energies of the Bloch waves, shows an important consistency between the standard, semiclassical theories of resistance [53] and the Landauer model.

In §3.3 a simple explanation of the length dependence of the relative importance of coherent and incoherent scattering and the crucial length scale l_ϕ at which the wavefunction loses its phase coherence is introduced. At length scales $L > l_\phi$ the time-reversal symmetry of the wavefunction is destroyed, which is then seen to provide a limit to the localisation length in disordered media [70]. These arguments also explain why theories of the resistance based on coherent scattering lead to results applicable to small samples ($L < l_\phi$) where little scattering occurs (ballistic transport). In such regions there is difficulty in defining an average resistance from the Landauer model [14–17], associated with the exponential length dependence of the resistance, which is the source of the ‘universal fluctuations’ of resistance for mesoscopic samples [20, 30].

The correct result for the composition of incoherent scatterers is shown to lead to the familiar, additive scaling of resistance, a result which has eluded workers since Landauer’s original paper of thirty years ago [1]. This separation of coherent from incoherent electron transport in the Landauer model is thus seen to neatly separate classical (macroscopic) from quantum interference behaviour [59].

In summary, the Landauer model is adequate for small ($L < l_\phi$), plane parallel

samples, where ballistic transport operates. In this region two novel interpretations of the effect of scatterers have been introduced. For macroscopic samples ($L \gg l_\phi$) however, incoherent scattering must be included, which unfortunately invalidates the Landauer model and the associated T -matrix representation. Here, a new theory based on multiple incoherent scattering is required: one candidate being the two-stream approximation of radiative transfer [80], which leads to a linear length scaling.

As a final note, the central role played by incoherent scattering in giving this macroscopic behaviour is obvious with hindsight. In an early paper [27], Landauer noted that the classical composition law requires the combination of probabilities rather than amplitudes, a result long associated with incoherence in the field of optics [81].

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